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INVERSE PROBLEMS IN THE DYNAMICS OF PARABOLIC SYSTEMS*

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In the context of an approach to inverse problems of dynamics /1, 2/, regularizing finite-stepped dynamical algorithms are proposed to determine the sources of disturbances in processes described by evolution equations.

1. Statement of the problem. We will begin with an intuitive description. Suppose that sources of a certain material are distributed in a domain Ω; their positions and strengths are unknown. The propagation of the material is observed for a certain time $T = [t_0, \vartheta]$ and at certain times $t_i \in T$, $i = 0, 1, \ldots, m$, the concentration of the material in the domain, represented by a scalar function $y(t_i, x)$, $x \in \Omega$, is measured. The result of the measurement is a quantity $\xi(t_i, x)$, $x \in \Omega$, which satisfies a mean-square estimate

$$\int_\Omega (\xi(t_i, x) - y(t_i, x))^2\, dx \leqslant h^2$$

An algorithm will be constructed which, during the process (on a real-time basis), utilizes the incoming information about the concentration of material to determine (an approximation to) the positions and strengths of the sources. The algorithm is regularizing in the sense that the output delivered may be improved by reducing the measurement errors and measuring the concentration at more frequent intervals.

The problem may be formulated rigorously as follows. Let us assume that at a time $t \in T$ the sources are concentrated in some unknown set $G(t) \subset \Omega$. The strength of a source at a point $x \in \Omega$ at time $t \in T$ is defined by a scalar quantity $f(t, x)$, of which only an a priori estimate $\beta_1 \leqslant f(t, x) \leqslant \beta_2$, $t \in T$, $x \in \Omega$, is known, where $\beta_1 = \text{const} > 0$, $\beta_2 = \text{const} \geqslant \beta_1$. The sets

$$S = \{(t, x):\ t \in T,\ x \in G(t)\},\ G(t)\ \ (t \in T)$$

and function f are assumed to be Lebesgue-measurable. The dynamics of the process is described by the boundary-value problem

$$\begin{gathered}\partial y/\partial t = A(t)\, y + \chi_{G(t)}(x)\, f(t, x) \text{ in } Q = T \times \Omega \\ \sigma_1 \partial y/\partial N + \sigma_2 y = 0 \text{ on } \Sigma = T \times \Gamma \\ y(t_0, x) = y_0(x) \text{ in } \Omega\end{gathered} \tag{1.1}$$

Here Ω is a bounded open connected set in Euclidean space R^n $(n \geqslant 1)$ with piecewise smooth boundary Γ (for the sequel it will suffice to assume that Ω is strictly Lipschitzian /4, p.30/; $\chi_B(\cdot)$ is the characteristic function of a set $B \subset \Omega$: $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$; $\partial y/\partial N$ is the outward conormal derivative; σ_1 and σ_2 are non-negative numbers, $\sigma_1 + \sigma_2 > 0$; $L_2(\Omega) \ni y_0$ is the initial distribution (at time $t = t_0$) of the concentration of the material over Ω; $A(t)$ is a coercive linear selfadjoint elliptic operator

$$A(t)\, y = \sum_{i, j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(t, x) \frac{\partial y}{\partial x_j}\right) - a(t, x)\, y$$

$$a_{ij} = a_{ji} \in L_\infty(Q),\ a \in L_\infty(Q),\ a \geqslant 0$$

With the parameters subjected to these restrictions, there exists a unique generalized solution $y = y(t, x)$, $t \in T$, $x \in \Omega$, to the boundary-value problem (1.1), and it is an element of the space $V_2^{1,0}(Q)$ /5/.

Let (Q^*, d) denote the semimetric space defined by letting Q^* by the set of all Lebesgue-measurable subsets of Q and d the semimetric $d(E_1, E_2) = \text{mes}(E_1 \Delta E_2)$, where $E_1 \Delta E_2$ is the

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symmetric distance of the sets E_1 and E_2 and mes denotes Lebesgue measure in R^{n+1}. Let P be the convex bounded closed subset of $L_2(\Omega)$ which contains all possible elements $\chi_B g$, where B may be any Lebesgue-measurable subset of Ω and g any function in $L_2(\Omega)$ such that, for almost all $x \in \Omega$, $\beta_1 \leqslant g(x) \leqslant \beta_2$. Any finite family $(\tau_i)_{i=0,\ldots,m}$, where $t_0 = \tau_0 < \ldots < \tau_m = \vartheta$, will be called a partition of T.

The problem is to design an algorithm which will generate an approximation to the set S and function f, on the assumption that at every instant of time $t \in T$ the concentration of material $y(t) = y(t, \cdot)$ in Ω can be measured, and the result of the measurement $\xi(t) = \xi(t, \cdot)$ is related to $y(t)$ through the inequality

$$\| \xi(t) - y(t) \|_{L_2(\Omega)} \leqslant h \tag{1.2}$$

A solution to this problem will be sought in the class of finite-stepped dynamical algorithms (FDAs) /2/. By a FDA we mean any triple

$$D = ((\tau_i)_{i=0,\ldots,m};\ (r_i)_{i=0,\ldots,m-1};\ (\rho_i)_{i=0,\ldots,m-1}) \tag{1.3}$$

where m is a natural number, $(\tau_i)_{i=0,\ldots,m}$ a partition of T, r_i is a map of $L_2(\Omega) \times L_2(\Omega)$ into P, and ρ_i is a map of $L_2(\Omega) \times L_2(\Omega)$ into $L_2(\Omega)$, $i = 0, \ldots, m-1$. For every FDA (1.3) and function $\xi: T \to L_2(\Omega)$, we define a (D, ξ)-realization to be a pair of elements $(u, E) \in L_2(T; L_2(\Omega)) \times Q^*$, formed by the following rule: $u(t) = r_i(\xi(\tau_i), z_i)$ for $t \in [\tau_i, \tau_{i+1})$ and $i = 0, \ldots, m-1$, $z(t_0) = \xi(t_0)$, $z_{i+1} = \rho_i(\xi(\tau_i), z_i)$ for $i = 0, \ldots, m-1$, $E = \{(t, x) \in T \times \Omega: u(t, x) \geqslant \beta_1/2\}$. The function ξ will be called the input to the algorithm and a (D, ξ)-realization its output.

The FDA operates as follows with elapsing time. Before the starting time t_0 it selects and records a partition $(\tau_i)_{i=0,\ldots,m}$, each point of which τ_i will serve as the starting time of the next step (cycle) of the computation. At time $t = \tau_i$, $i = 0, \ldots, m-1$, the algorithm receives information $\xi(\tau_i)$ (the measured value of the function $y(\tau_i)$); on the basis of this information and the value z_i of the auxiliary variable z ($z_0 = \xi(t_0)$) up to time $t = \tau_{i+1}$, the algorithm determines a new value z_{i+1} of the auxiliary variable according to the rule ρ_i, an element $u_i = r_i(\xi(\tau_i), z_i) \in P$ according to the rule r_i, and a set $E_i = [\tau_i, \tau_{i+1}) \times \{x \in \Omega: u_i(x) \geqslant \beta_1/2\}$. Up to the terminal time ϑ the algorithm computes a (D, ξ)-realization (u, E): $u(t) = u_i$ for $t \in [\tau_i, \tau_{i+1})$ and $i = 0, \ldots, m-1$, $E = E_0 \cup \ldots \cup E_{m-1}$. The set E is the required approximation to S, and the function u to f.

Let Ξ_h $(h > 0)$ denote the set of all functions $\xi: T \to L_2(\Omega)$ which satisfy (1.2) for all $t \in T$. A family of FDAs $(D_h)_{h>0}$ is said to be regularizing if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in (0, \delta)$ and any function $\xi \in \Xi_h$ the (D, ξ)-realization (u, E) satisfies the inequalities

$$\| u - f \|_{L_2(S)} < \varepsilon,\ d(E, S) < \varepsilon$$

A function $\nu: [0, \infty) \to [0, \infty)$ is called an accuracy for the family $(D_h)_{h>0}$ if, for any $h > 0$ and $\xi \in \Xi_h$,

$$\| u - f \|_{L_2(S)} \leqslant \nu(h),\ d(E, S) \leqslant \nu(h),$$

where (u, E) is a (D_h, ξ)-realization. A family of FDAs $(D_h)_{h>0}$ is regularizing if and only if it has an accuracy $\nu(\cdot)$ and moreover $\nu(h) \to 0$ as $h \to 0$. Obviously, a regularizing family FDA provides a solution to the problem as formulated above. It will be used to solve our reconstruction problem in the following way. Given a preassigned error h, choose a suitable FDA D_h from the family and use it to determine the positions and strengths of the sources (the operation of any such algorithm was described above). Our construction guarantees that, the smaller h, the more accurately will the (D_h, ξ)-realization (u, E) delivered at the output of the algorithm approximate the pair (f, S) in the sense of the metric of the space $L_2(S) \times Q^*$.

2. ***The construction of a regularizing family of FDAs.*** Underlying the following construction of a regularizing family of FDAs are certain ideas from /1, 2/. Define a family of FDAs $(D_h)_{h>0}$ by the conditions:

$$D_h = ((\tau_i^h)_{i=0,\ldots,m};\ (r_i^h)_{i=0,\ldots,m_h-1};\ (\rho_i^h)_{i=0,\ldots,m_h-1}) \tag{2.1}$$

where for any $h > 0$, $i = 0, \ldots, m_h - 1$, $\xi \in L_2(\Omega)$, $z \in L_2(\Omega)$ the element $r_i^h(\xi, z)$ is a minimum point of the quadratic functional

$$\Phi(u) = 2\langle z - \xi, u \rangle_{L_2(\Omega)} + \alpha(h) \| u \|^2_{L_2(\Omega)}$$

on the set P (such a point exists and is unique), $\rho_i^h(\xi, z) = w(\tau_{i+1}^h)$, where w is a generalized solution in the space $V_2^{1,0}([\tau_i^h, \tau_{i+1}^h] \times \Omega)$ of the boundary-value problem

$$\partial w/\partial t = A(t) w + r_i^h(\xi, z) \text{ in } [\tau_i^h, \tau_{i+1}^h] \times \Omega \tag{2.2}$$

$$\sigma_1 \partial w/\partial N + \sigma_2 w = 0 \text{ on } [\tau_i^h, \tau_{i+1}^h] \times \Gamma$$

$$w(\tau_i^h) = z \text{ in } \Omega$$

Here α: $[0, \infty) \to [0, \infty)$ is some fixed auxiliary function.

Theorem 2.1. Let the function α and the quantity $\Delta(h) = \max\{\tau_{i+1}^h - \tau_i^h : i = 0, \ldots, m_h - 1\}$ be such that $\alpha(h) \to 0$, $\Delta(h) \to 0$, $h/\alpha(h) \to 0$, $\sqrt{\Delta(h)}/\alpha(h) \to 0$ as $h \to 0$. Then there exists $h_0 > 0$ such that for any $h \in (0, h_0)$, the sets E_h in the (D_h, ξ)-realizations (u_h, E_h) are not empty and the family of FDAs (2.1) is regularizing.

Note that $\alpha(h)$ and $\Delta(h)$ which satisfy the conditions of Theorem 3.1 may indeed be chosen: it is sufficient for example, to define $\alpha(h) = h^\gamma$ $(0 < \gamma < 1)$, $\Delta(h) = h^{3\gamma}$. The proof of Theorem 2.1 is analogous to the proofs of the corresponding propositions in /2, 3/, being based on the following facts:

1) if $u \in L_2(Q)$, $B \in Q^*$, $E = \{(t, x) \in T \times \Omega: u(t, x) \geqslant \beta_1/2\}$, then

$$\| u - \chi_B f \|^2_{L_2(Q)} \geqslant \left(\frac{\beta_1}{2}\right)^2 d(E, B)$$

2) if $\{h_k\} \subset (0, \infty)$, $\{\xi_k\}$, $\{(u_k, E_k)\}$ are sequences such that $h_k \to 0$, $\xi_k \in \Xi_k$, (u_k, E_k) are (D_k, ξ_k)-realizations $(\Xi_k = \Xi_h, D_k = D_h$ for $h = h_k$, $k = 1, 2, \ldots)$, then under the assumptions of Theorem 2.1

$$\| u_k - \chi_S f \|_{L_2(Q)} \to 0, \quad d(E_k, S) \to 0$$

We will now provide an intuitive description of the sequence of operations to be performed when reconstructing the sources and their strengths in accordance with Theorem 2.1.

One first selects $\alpha(h)$, $\Delta(h)$ and partitions $(\tau_i^h)_{i=0,\ldots,m_h}$ such that $\alpha(h) \to 0$, $\Delta(h) \to 0$, $h/\alpha(h) \to 0$, $\sqrt{\Delta(h)}/\alpha(h) \to 0$ as $h \to 0$. On receiving (before a time t_0) a value for the error h, it is recorded, as are the value of $\alpha(h)$ and the partition $(\tau_i^h)_{i=0,\ldots,m_h}$ of T. Beginning at time t_0, in each of the time intervals $[\tau_i^h, \tau_{i+1}^h)$, $i = 0, \ldots, m_h - 1$ in succession, the values u_i of the auxiliary function u are computed $(u(t) = u_i$ for $t \in [\tau_i^h, \tau_{i+1}^h)$ and $i = 0, \ldots, m_h - 1)$, the part E_i of the "trajectory" of source positions E $(E = E_0 \cup \ldots \cup E_k$, $k = m_h - 1)$ is determined, the values of the auxiliary variable z are recalculated, replacing z_i with z_{i+1}, according to the rules: u_i is the minimum point on P of the quadratic functional

$$u \to 2\langle z_i - \xi(\tau_i^h), u\rangle_{L_2(\Omega)} + \alpha(h) \| u \|^2_{L_2(\Omega)}$$
$$E_i = [\tau_i^h, \tau_{i+1}^h) \times \{x \in \Omega: u_i(x) \geqslant \beta_1/2\}$$
$$z_{i+1} = w(\tau_{i+1}^h)$$

where w is the solution of the boundary-value problem (2.2) with $z = z_i$ and $r_i^h(\xi, z) = u_i$ (we define $z_0 = \xi(t_0)$). When this is done, u will approximate $f\chi_S$ in the metric of $L_2(Q)$, and E will approximate S in the semimetric d.

3. ***Estimate of the accuracy of a FDA.*** We now describe one modification of the family of FDAs (2.1), (2.2) and exhibit the form of the appropriate accuracy ν. The new family of FDAs is defined in the same way as the family (2.1), (2.2), but so as to satisfy the condition

$$w(\tau_{i+1}^h) = z + \int_{\tau_i^h}^{\tau_{i+1}^h} [A(\tau)\xi + r_i^h(\xi, z)]\, d\tau \tag{3.1}$$

Put $V = W_2^2(\Omega) \cap \overset{0}{W}{}_2^1(\Omega)$ if $\sigma_1 = 0$, $V = W_2^2(\Omega)$ if $\sigma_1 \neq 0$.

Condition 3.1. 1) For every $t \in T$ the operator $A(t)$ is linear and bounded from V to $L_2(\Omega)$, and the norms of these operators are uniformly bounded, i.e., there exists a number $c_0 > 0$ such that for every $t \in T$ one has $\| A(t) \|_{V \to L_2(\Omega)} \leqslant c_0$; 2) the solution of the boundary-value problem (1.1) is uniformly continuous as a map $T \to V$, i.e., there exists a function $\sigma: [0, \infty) \to [0, \infty)$ such that $\sigma(t) \to 0$ as $t \to 0$ and $\| y(t_1) - y(t_2) \|_V \leqslant \sigma(|t_1 - t_2|)$ for any $t_1, t_2 \in T$; 3) at every $t \in T$ the measurement $\xi(t) \in V$ and current state $y(t)$ of system (1.1) satisfy the inequality $\| \xi(t) - y(t) \|_V \leqslant h$; 4) the function $v: T \ni t \to \chi_{G(t)}(\cdot) f(t, \cdot) \in L_2(\Omega)$ has a bounded variation on T.

Let

$$\Delta = \Delta(h), \ \gamma_P = \max\{\|u\|_{L_2(\Omega)}: u \in P\}$$
$$\varepsilon(h) = h^2 + (\vartheta - t_0)\gamma(h + \sqrt{\Delta})$$
$$\delta(h) = [\varepsilon(h) + 2\alpha(h)(\vartheta - t_0)\gamma_P^2]^{1/2}$$

where γ is a positive number, determined from the known parameters of the boundary-value problem (1.1), such that for any $h > 0$, $t \in T$, $\xi \in \Xi_h$

$$\|w(t) - y(t)\|^2_{L_2(\Omega)} + \alpha(h)\int_{t_0}^{t}(\|u(\tau)\|^2_{L_2(\Omega)} - \|v(\tau)\|^2_{L_2(\Omega)})\,d\tau \leqslant \varepsilon(h)$$
$$w(t) = \xi(t_0) + \int_{t_0}^{t}[A(\tau)\bar{\xi}(\tau) + u(\tau)]\,d\tau$$

where (u, E) is a (D_h, ξ)-realization, $\bar{\xi}(t) = \xi(\tau_i^h)$ for $t \in [\tau_i^h, \tau_{i+1}^h)$, $i = 0, \ldots, m_h - 1$.

Theorem 3.1. If Condition 3.1 is satisfied the accuracy ν of the family of FDAs (2.1), (3.1) has the form

$$\nu(h) = [\varepsilon(h)/\alpha(h) + 2(\operatorname{var} v + \gamma_v)(\delta(h) + c_0(\vartheta - t_0)h + \sigma(\Delta(h)))]^{1/2}$$

and $\nu(h) \to 0$, if $\alpha(h) \to 0$, $\Delta(h) \to 0$, $h/\alpha(h) \to 0$, $\sqrt{\Delta(h)}/\alpha(h) \to 0$ as $h \to 0$.

The proof of Theorem 3.1 is analogous to that of the parallel fact in /6/. Parts 1 and 2 of Condition 3.1 are satisfied for regular domains Ω and operators $A(t)$ with sufficiently smooth coefficients (see, e.g., /4, Chap.3, Sects.8, 9/; /5, Chap.3, Sect.6/), while parts 3 and 4 impose restrictions on the current measurements of the state of the system and the way in which the sources may vary.

4. *Remarks.* 1°. The family of FDAs (2.1) may also be used to determine sources when the minimum of the functional $\Phi(u)$ and values of the maps ρ_i^h are computed with certain errors. For example, instead of the exact values of $r_i^h(\xi, z)$ and $\rho_i^h(\xi, z)$ one might use approximations $\bar{r}_i^h(\xi, z)$ and $\bar{\rho}_i^h(\xi, z)$, provided that these satisfy the inequalities

$$|\Phi(r_i^h(\xi, z)) - \Phi(\bar{r}_i^h(\xi, z))| \leqslant \varkappa_1(h)(\tau_{i+1}^h - \tau_i^h)$$
$$\|\rho_i^h(\xi, z) - \bar{\rho}_i^h(\xi, z)\|_{L_2(\Omega)} \leqslant \varkappa_2(h)(\tau_{i+1}^h - \tau_i^h)$$

where $\varkappa_1$ and $\varkappa_2$ are certain fixed auxiliary functions $[0, \infty) \to [0, \infty)$ such that $\varkappa_1(h) \to 0$ and $\varkappa_2(h) \to 0$ as $h \to 0$. Under these conditions the family of FDAs (2.1) is still regularizing if, in addition to the conventions from Theorem 2.1, one also imposes coordination conditions such as $\varkappa_1(h)/\alpha(h) \to 0$ and $\varkappa_2(h)/\alpha(h) \to 0$ as $h \to 0$.

2°. Similar techniques can be used to reconstruct sources on the basis of points measurements of the concentration in Ω. Indeed, suppose that a suitable grid is imposed on Ω, dividing it into disjoint cells Ω_i, $i \in I$, and that at successive instants of time $t \in T$ the concentrations are measured at various points $x_i \in \Omega_i$, $i \in I$, the results being scalar quantities $\xi_i(t)$, $i \in I$, such that

$$\left|\xi_i(t) - \frac{1}{\operatorname{mes}(\Omega_i)}\int_{\Omega_i} y(t, x)\,dx\right| \leqslant h, \ t \in T, \ i \in I$$

Let $\xi(t) = \xi(t, \cdot)$ denote a step function interpolating the grid function $\xi_i(t)$, $i \in I$, in Ω: $\xi(t, x) = \xi_i(t)$ if $x \in \Omega_i$ and $i \in I$. We have

$$\|\xi(t) - y(t)\|_{L_2(\Omega)} \leqslant c(h + \mu(\varkappa)), \quad t \in T, \ c = \text{const} > 0$$

where $\varkappa$ is the mesh of the partition of Ω, μ is some function $[0, \infty) \to [0, \infty)$ such that $\mu(\varkappa) \to 0$ as $\varkappa \to 0$ (c and μ are determined from the known parameters of problem (2.1)). Then the family of FDAs (2.1) is still regularizing, provided that the function $\xi(t, \cdot)$ is taken as the result of measuring the concentration $y(t, \cdot)$ at time $t \in T$, and besides the conditions of Theorem 2.1 it is also assumed that

$$\varkappa(h) \to 0, \quad \mu(\varkappa(h))/\alpha(h) \to 0 \ \text{as} \ h \to 0.$$

3°. Let us see how to go about the reconstruction of sources in the Hausdorff metric. The following assertion is true: if (u_h, E_h) is a (D_h, ξ)-realization and as $h \to 0$ one has $u_h \to \chi_{G(\cdot)} f(\cdot)$ in the supnorm, then $E_h \to S$ in the Hausdorff metric. With suitable assumptions, the set of realizations $\{u_h\}$ may prove to be precompact in $C(Q)$, and then, if the realizations are convergent in $L_2(Q)$, they are uniformly convergent.

4°. Analogous results have been obtained for certain classes of quasilinear parabolic systems, described by the equations

$$\frac{\partial y}{\partial t}=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}a_i(t,x,y,y_x)-a(t,x,y,y_x)+\chi_{G(t)}f(t,x)\ \text{in } Q$$

$$\sigma_1\frac{\partial y}{\partial N}+\sigma_2 y=g(t,x)\ \text{ on }\ \Sigma$$

$$y(t_0,\ x)=y_0(x),\ \text{in}\ \Omega$$

5°. The algorithms obtained above may also be modified to determine the sources on the boundary:

$$\sigma_1\,\partial y/\partial N+\sigma_2 y=\chi_{\Gamma(t)}\,g(t,\ x)$$

where $\Gamma(t),\ t\in T$, are the unknown positions of the sources on the boundary, $g(t,\ x),\ (t,\ x)\in\Sigma$, is the unknown strength of the sources.

5. *Example.* Let us consider the problem of reconstructing the distribution of sources in a two-dimensional domain $\Omega=(0,\ l_1)\times(0,\ l_2)$, assuming that the dynamical process is governed by the boundary-value problem

$$\frac{\partial y}{\partial t}=a\frac{\partial^2 y}{\partial x_1^2}+b\frac{\partial^2 y}{\partial x_2^2}+\chi_{G(t)}f(t,x)\ \text{in } Q$$

$$y=0\ \text{on}\ \Sigma$$

$$y(t_0,x_1,x_2)=y_0(x_1,x_2)\quad\text{in}\quad\Omega$$

Numerical computations have been carried out for various parameter values and configurations of the sets $G(t),\ t\in T$. The evolution of the dynamical system and of the auxiliary model were implemented by means of an explicit difference scheme with uniform time step-size and uniform spatial mesh along the x_1 and x_2 axes. The simulation shows that as early as the second or third step one begins to get a stable picture of the reconstructed source positions. The required set is determined by rectangles with centres at the mesh points and sides equal to the mesh lengths in the x_1 and x_2 directions.

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